J. Fluid Mech. (1993), *vol.* 249, *pp.* 121–133

Nonlinear evolution of surface gravity waves over an uneven bottom

By Y. MATSUNO

Department of Physics, Faculty of Liberal Arts, Yamaguchi University, Yamaguchi 753, Japan

(Received 6 August 1992 and in revised form 15 October 1992)

A unified theory is developed which describes nonlinear evolution of surface gravity waves propagating over an uneven bottom in the case of two-dimensional incompressible and inviscid fluid of arbitrary depth. Under the assumptions that the bottom of the fluid has a slowly varying profile and the wave steepness is small, a system of approximate nonlinear evolution equations (NEEs) for the surface elevation and the horizontal component of surface velocity is derived on the basis of a systematic perturbation method with respect to the steepness parameter. A single NEE for the surface elevation is also presented. These equations are expressed in terms of original coordinate variables and therefore they have a direct relevance to physical systems. Since the formalism does not rely on the often used assumptions of shallow water and long waves, the NEEs obtained are uniformly valid from shallow water to deep water and have wide applications in various wave phenomena of physical and engineering importance. The shallow- and deep-water limits of the equations are discussed and the results are compared with existing theories. It is found that our theory includes as specific cases almost all approximate theories known at present.

1. Introduction

The nonlinear dynamics of surface gravity waves on fluid have been studied extensively since the pioneering work of Stokes (1849). Various types of approximate nonlinear evolution equations (NEEs) have been derived according to the situations under consideration (see, for instance Whitham 1974; Mei 1989). Equations thus obtained may be divided into two groups, namely shallow-water theories and deep-water theories. The Boussinesq and the Korteweg–de Vries theories belong to the former class whereas the Stokes theory is a typical example of the latter. The finite-depth analogue of these theories has been established quite recently (Matsuno 1992).

Most model equations proposed until now are concerned with the nonlinear evolution of surface gravity waves on fluid of *uniform* depth. However, because of its practical importance, several attempts have been made to take into account of the effects of an uneven bottom (Mei & Méhauté 1966; Peregrine 1967; Madsen & Mei 1969; Grimshaw 1970; Kakutani 1971; Johnson 1973; Wu 1981). These works are mainly based on the assumptions of both shallow water and long waves and hence the range of applicability is severely limited. Recently, Radder (1992) developed an explicit Hamiltonian formulation of surface waves on fluid of finite depth and compared it with existing theories (Broer 1974, 1975). He also discussed the limiting cases of both shallow and deep fluids.

The purpose of this paper is to develop a unified theory of nonlinear surface gravity waves over an uneven bottom with *finite* depth. A new method presented here is a generalization of the theory of surface gravity waves on fluid of uniform depth formulated by the author (Matsuno 1992).

We consider the two-dimensional irrotational flow of an incompressible and inviscid fluid. The bottom profile is assumed to be a slowly varying function of the horizontal coordinate. Although this assumption is not essential in developing the theory, it considerably simplifies the analysis. The formalism in this paper relies only on the assumption of small wave steepness which implies that we focus on waves of small but finite amplitude. The assumptions such as shallow water and long waves are not used in deriving NEEs and this fact provides much greater flexibility in dealing with specific physical systems.

In §2, the governing equation of fluid motion is described with the boundary conditions in appropriate dimensionless form and then they are transformed into those for a fluid region with uniform depth by using a conformal mapping. The solutions of the transformed equations are constructed explicitly in §3. In §4, by employing a systematic perturbation method with respect to the steepness parameter, we first derive approximate NEEs in terms of transformed variables and then rewrite them in original physical variables. We thus obtain a system of equations for the surface elevation and the horizontal component of surface velocity. A single equation for the surface elevation is also presented. In §5, equations arising from both shallow- and deep-water limits of approximate equations are discussed and they are compared with existing theories. Section 6 is devoted to conclusions.

2. Basic equations and their transformations

2.1. *Basic equations*

First, we describe the equation of motion of fluid together with the boundary conditions. All the variables are non-dimensionalized appropriately as shown later. Under the assumption mentioned in §1, the fluid motion is governed by the Laplace equation

$$\delta^2\phi_{xx}+\phi_{yy}=0 \quad \text{in} \quad -\infty<x<\infty, \quad b<y<\alpha h, \tag{2.1}$$

with the boundary conditions

$$h_t+\kappa\epsilon\phi_x h_x=\frac{\kappa}{\delta}\phi_y \quad \text{on} \quad y=\alpha h, \tag{2.2}$$

$$\phi_t+\frac{\kappa\epsilon}{2\delta^2}(\delta^2\phi_x^2+\phi_y^2)+\alpha^{-1}(y-y_0)=0 \quad \text{on} \quad y=\alpha h \quad (y_0\text{: const.}), \tag{2.3}$$

$$\delta^2\phi_x b_x=\phi_y \quad \text{on} \quad y=b. \tag{2.4}$$

Here $\phi=\phi(x,y,t)$ is the velocity potential, $h=h(x,t)$ is the surface elevation, $b=b(x)$ is the profile of the bottom and the subscripts x, y and t appended to ϕ, h and b denote partial differentiations. Equation (2.1) stems from the assumption of irrotational flow of an incompressible fluid. Equations (2.2) and (2.3) represent the kinematic and dynamic conditions on the free surface, respectively, while the condition (2.4) comes from the fact that the flow direction must be that of the bottom since the fluid is inviscid. The dimensional quantities with tildes are related to the corresponding dimensionless ones by the relations

$$\left.\begin{aligned}&\tilde{x}=lx, \quad \tilde{y}=h_0 y, \quad \tilde{t}=(l/c_0)t,\\ &\tilde{\phi}=(gla/c_0)\phi, \quad \tilde{h}=ah, \quad \tilde{b}=h_0 b,\end{aligned}\right\} \tag{2.5}$$

where l, a and c_0 are characteristic scales of length (wavelength in a periodic wave), amplitude and velocity of the wave, respectively, h_0 is a typical vertical lengthscale which may be taken as an undisturbed fluid depth at $x = 0$ and g is the acceleration due to gravity. The surface elevation h is measured from the undisturbed fluid surface which is chosen to be $y = 0$ in the present case. The dimensionless parameters ϵ, α and δ are defined by

$$\epsilon = a/l, \quad \alpha = a/h_0, \quad \delta = h_0/l. \tag{2.6}$$

These parameters are connected to each other by the relation $\epsilon = \alpha\delta$. The ϵ is called the steepness parameter. The c_0 is given by $c_0 = (gl/\kappa)^{\frac{1}{2}}$ where κ is assumed to be δ^{-1} in the shallow-water limit $\delta \to 0$ and 1 in the deep-water limit $\delta \to \infty$ in accordance with the phase velocity of linear surface gravity waves on fluid of uniform depth h_0. The effect of surface tension has been neglected to simplify the analysis, but it can be included without any difficulty.

The bottom of the fluid is assumed to have a slowly varying profile and it may be expressed in the form

$$b(x) = -1 + B(\alpha x), \tag{2.7}$$

where B represents the bump on the bottom and $B(0) = 0$ by the definition of the present configuration. This means that the measure of the changing depth is chosen to be the same order as that for the surface elevation. The magnitude of B itself may be of the order of unity, however.

2.2. *Transformations of basic equations*

In order to apply the method developed by Matsuno (1992), it is necessary to transform the basic equations into those for a fluid region with a flat bottom. The corresponding coordinate transformation is well known and it may be represented in the form (Woods 1961; Byatt-Smith 1971)

$$y(\xi, \eta) = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin \pi\eta}{\cos \pi\eta + \cosh \pi\xi'} \bar{b}(\xi - \delta\xi')\,\mathrm{d}\xi', \tag{2.8a}$$

$$x(\xi, \eta) = \int_0^{\xi} \frac{\partial y(\xi', \eta)}{\partial \eta}\,\mathrm{d}\xi', \tag{2.8b}$$

where $\bar{b}(\xi) = b(x)$ and the fluid region $-\infty < x < \infty$, $b(x) < y < \alpha h(x, t)$ has been mapped conformally into the region $-\infty < \xi < \infty$, $-1 < \eta < \alpha\bar{h}(\xi, t)$ with $\bar{h}$ being the surface elevation in the (ξ, η)-plane. It now follows from the differential relations $\mathrm{d}x = x_\xi\,\mathrm{d}\xi + x_\eta\,\mathrm{d}\eta$ and $\mathrm{d}y = y_\xi\,\mathrm{d}\xi + y_\eta\,\mathrm{d}\eta$ that

$$x_\xi \xi_x + x_\eta \eta_x = 1, \quad x_\xi \xi_y + x_\eta \eta_y = 0, \tag{2.9a}$$

$$y_\xi \xi_x + y_\eta \eta_x = 0, \quad y_\xi \xi_y + y_\eta \eta_y = 1. \tag{2.9b}$$

On the other hand, since the transformation (2.8) is conformal in the sense that $x + \mathrm{i}\delta y$ is an analytic function of $\delta^{-1}\xi + \mathrm{i}\eta$, the following Cauchy–Riemann relations result as an equivalent condition:

$$x_\xi = y_\eta, \quad x_\eta = -\delta^2 y_\xi. \tag{2.10}$$

Consequently, x and y are found to satisfy the same Laplace equation as (2.1). By combining (2.9) and (2.10), we obtain the important formulae which will often be used in this paper

$$\xi_x = y_\eta/J, \quad \xi_y = \delta^2 y_\xi/J, \quad \eta_x = -y_\xi/J, \quad \eta_y = y_\eta/J, \tag{2.11a}$$

where
$$J = \delta^2 y_\xi^2 + y_\eta^2, \tag{2.11b}$$
is the Jacobian of the transformation (2.8).

Equations (2.1)–(2.4) are then transformed into the following forms:

$$\delta^2 \bar{\phi}_{\xi\xi} + \bar{\phi}_{\eta\eta} = 0 \quad \text{in} \quad -\infty < \xi < \infty, \quad -1 < \eta < \alpha\bar{h}, \tag{2.12}$$

$$\bar{h}_t + \frac{\kappa\delta}{J y_\eta}(y_\eta \bar{\phi}_\xi - y_\xi \bar{\phi}_\eta)\frac{y_\xi + \alpha y_\eta \bar{h}_\xi}{y_\eta - \alpha\delta^2 y_\xi \bar{h}_\xi} = \frac{\kappa}{\delta J y_\eta}(\delta^2 y_\xi \bar{\phi}_\xi + y_\eta \bar{\phi}_\eta) \quad \text{on} \quad \eta = \alpha\bar{h}, \tag{2.13}$$

$$\bar{\phi}_t + \frac{\kappa\epsilon}{2\delta^2 J}(\delta^2 \bar{\phi}_\xi^2 + \bar{\phi}_\eta^2) + \alpha^{-1}[y(\xi,\eta) - y_0] = 0 \quad \text{on} \quad \eta = \alpha\bar{h}, \tag{2.14}$$

$$\bar{\phi}_\eta = 0 \quad \text{on} \quad \eta = -1, \tag{2.15}$$

where $\bar{\phi}(\xi,\eta,t) = \phi(x,y,t)$ and the surface elevations in original and transformed systems are related to each other by

$$h = -\frac{1}{2\alpha}\int_{-\infty}^{\infty} \frac{\sin \pi\alpha\bar{h}}{\cos \pi\alpha\bar{h} + \cosh \pi\xi'} \bar{b}(\xi - \delta\xi')\,\mathrm{d}\xi'. \tag{2.16}$$

Equation (2.12) follows immediately due to the conformal property of (2.8). This can also be confirmed easily by a direct calculation. To derive (2.13), we first observe that on the fluid surface

$$h_t = y_\eta \bar{h}_t, \qquad \alpha h_x = \frac{y_\xi + \alpha y_\eta \bar{h}_\xi}{y_\eta - \alpha\delta^2 y_\xi \bar{h}_\xi}. \tag{2.17a, b}$$

Substituting (2.17) and the relations

$$\phi_x = \xi_x \bar{\phi}_\xi + \eta_x \bar{\phi}_\eta = (y_\eta \bar{\phi}_\xi - y_\xi \bar{\phi}_\eta)/J, \tag{2.18a}$$
$$\phi_y = \xi_y \bar{\phi}_\xi + \eta_y \bar{\phi}_\eta = (\delta^2 y_\xi \bar{\phi}_\xi + y_\eta \bar{\phi}_\eta)/J, \tag{2.18b}$$

into (2.3) yields (2.13), while (2.14) follows from (2.11*b*) and (2.18). To verify (2.15), we first note that on the bottom of fluid the relation $\eta(x, b(x)) = -1$ holds. Differentiation of it with respect to x yields $\eta_x + \eta_y b_x = 0$. Substituting this equation together with (2.11) and (2.18) into (2.4), we arrive at (2.15). It should be stressed that the transformed equations are valid as long as the Jacobian (2.11*b*) does not vanish.

3. Solutions of transformed equations

Once the basic equations have been transformed into those for the flat bottom, solutions can be constructed following the procedure developed by the author (Matsuno 1992). In this section we shall summarize the method of solution and present explicit solutions.

We take the solution of (2.12) which satisfies the bottom boundary condition (2.15) of the form

$$\bar{\phi} = -\mathrm{i}[f_+(\xi - \mathrm{i}\delta\eta, t) - f_-(\xi + \mathrm{i}\delta\eta, t)], \tag{3.1}$$

where $f_+(\zeta,t)\,(f_-(\zeta,t))$ is an analytic function of $\zeta(=\xi+\mathrm{i}\eta)$ in the strip $0 < \eta < 2\delta\,(-2\delta < \eta < 0)$ and given explicitly by the integral representation

$$f_\pm(\zeta,t) = \pm\frac{1}{4\mathrm{i}\delta}\int_{-\infty}^{\infty} \coth[\pi(\xi' - \zeta)/2\delta] f(\xi',t)\,\mathrm{d}\xi'. \tag{3.2}$$

Here f is an arbitrary real function defined appropriately on the real axis. If we take the boundary values of $f_{\pm}$ when $\eta \to \pm 0$, we obtain the important relations

$$f_{\pm}(\xi \pm \mathrm{i}0, t) = \tfrac{1}{2}(1 \mp \mathrm{i}T) f(\xi, t), \tag{3.3a}$$

where the integral operator T is defined by

$$Tf(\xi, t) = \frac{1}{2\delta} P \int_{-\infty}^{\infty} \coth\left[\pi(\xi' - \xi)/2\delta\right] f(\xi', t)\, \mathrm{d}\xi'. \tag{3.3b}$$

The symbol P in front of the integral sign denotes the Cauchy principal value integral. It readily follows from (3.3) that

$$f_{+}(\xi + \mathrm{i}0, t) + f_{-}(\xi - \mathrm{i}0, t) = f(\xi, t), \tag{3.4a}$$

$$f_{+}(\xi + \mathrm{i}0, t) - f_{-}(\xi - \mathrm{i}0, t) = -\mathrm{i}Tf(\xi, t). \tag{3.4b}$$

If we use (3.1) and the relation $\epsilon = \alpha\delta$, we can evaluate the derivatives of the velocity potential on the free surface as follows:

$$\bar{\phi}_{\xi}|_{\eta = \alpha \bar{h}} = -\mathrm{i}[f_{+,\xi}(\xi - \mathrm{i}\epsilon\bar{h}, t) - f_{-,\xi}(\xi + \mathrm{i}\epsilon\bar{h}, t)], \tag{3.5a}$$

$$\bar{\phi}_{\eta}|_{\eta = \alpha \bar{h}} = -\delta[f_{+,\xi}(\xi - \mathrm{i}\epsilon\bar{h}, t) + f_{-,\xi}(\xi + \mathrm{i}\epsilon\bar{h}, t)], \tag{3.5b}$$

$$\bar{\phi}_{t}|_{\eta = \alpha \bar{h}} = -\mathrm{i}[f_{+,t}(\xi - \mathrm{i}\epsilon\bar{h}, t) - f_{-,t}(\xi + \mathrm{i}\epsilon\bar{h}, t)]. \tag{3.5c}$$

Substitution of (2.8) and (3.5) into (2.13) and (2.14) yields the *exact* system of NEEs for $\bar{h}$ and f.

4. Approximate equations

Since the system of equations obtained in §3 is intractable as it stands, we must introduce some approximations to simplify the equations. For the purpose, we first note that in the case of fluid with finite depth, the parameters δ and κ may be taken to be of the order of unity whereas the steepness parameter ϵ is assumed to be small compared to unity. Except for the profile of the bottom of the fluid (see (2.7)), this is the only assumption used in the present theory. The problem under consideration now reduces to the expansion of various quantities in power series of ϵ or in α by virtue of the relation $\epsilon = \alpha\delta$. In this section, we derive the NEEs correct up to $O(\epsilon)$. Extensions to higher-order equations can be made straightforwardly but with tedious calculations. We first consider the equations with transformed variables and then rewrite them with the original physical variables.

4.1. *Equations with transformed variables*

If we expand (3.5) in powers of ϵ and use (3.4), we obtain the first two terms of the expansions

$$\bar{\phi}_{\xi}|_{\eta = \alpha \bar{h}} = -Tf_{\xi} - \epsilon\bar{h}f_{\xi\xi} + O(\epsilon^2), \tag{4.1a}$$

$$\bar{\phi}_{\eta}|_{\eta = \alpha \bar{h}} = -\delta[f_{\xi} - \epsilon\bar{h}Tf_{\xi\xi} + O(\epsilon^2)], \tag{4.1b}$$

$$\bar{\phi}_{t}|_{\eta = \alpha \bar{h}} = -Tf_{t} - \epsilon\bar{h}f_{\xi t} + O(\epsilon^2). \tag{4.1c}$$

At this stage it is convenient to introduce the horizontal component of the surface velocity:

$$\bar{u} = \bar{\phi}_{\xi}|_{\eta = \alpha \bar{h}}. \tag{4.2}$$

Then f in (4.1a) can be solved iteratively in terms of $\bar{u}$ as

$$f_{\xi} = -\tilde{T}\bar{u} + \epsilon\tilde{T}(\bar{h}\tilde{T}\bar{u}_{\xi}) + O(\epsilon^2), \tag{4.3a}$$

where the operator $\tilde{T}$ is the inverse of T, namely $T\tilde{T} = \tilde{T}T = I$ and it is given explicitly by

$$\tilde{T}\bar{u}(\xi,t) = -\frac{1}{2\delta}P\int_{-\infty}^{\infty}\frac{\bar{u}(\xi',t)}{\sinh\left[\pi(\xi'-\xi)/2\delta\right]}\mathrm{d}\xi'. \tag{4.3b}$$

Substitution of (4.3) into (4.1 b) and the ξ-derivative of (4.1 c) yields

$$\bar{\phi}_\eta|_{\eta=\alpha\bar{h}} = -\delta[-\tilde{T}\bar{u}+\epsilon\{\bar{h}\bar{u}_\xi+\tilde{T}(\bar{h}\tilde{T}\bar{u}_\xi)\}+O(\epsilon^2)], \tag{4.4}$$

$$(\bar{\phi}_t|_{\eta=\alpha\bar{h}})_\xi = \bar{u}_t+\epsilon(\bar{h}_\xi\tilde{T}\bar{u}_t-\bar{h}_t\tilde{T}\bar{u}_\xi)+O(\epsilon^2). \tag{4.5}$$

Next we derive the approximate expressions for y_ξ, y_η and J on the free surface $\eta = \alpha\bar{h}$. It readily follows from (2.8) and (2.11) that

$$y_\xi = -\alpha\bar{b}_\xi\bar{h}+O(\alpha^3), \quad y_\eta = -\bar{b}+O(\alpha^2), \tag{4.6a, b}$$

$$J = \bar{b}^2+O(\alpha^2), \tag{4.7}$$

where

$$\bar{b}(\xi) = -1+\bar{B}(\alpha\xi). \tag{4.8}$$

Note that $\bar{b}_\xi = \alpha\bar{B}'(\alpha\xi) = O(\alpha)$ where the prime denotes differentiation with respect to $\alpha\xi$ so that y_ξ turns out to be $O(\alpha^2)$. Substituting (4.2) and (4.4)–(4.7) into (2.13) and the ξ-derivative of (2.14), we obtain a system of equations for $\bar{h}$ and $\bar{u}$:

$$\bar{h}_t-\frac{\kappa}{\bar{b}^2}\tilde{T}\bar{u}+\frac{\kappa\epsilon}{\bar{b}^2}[(\bar{u}\bar{h})_\xi+\tilde{T}(\bar{h}\tilde{T}\bar{u}_\xi)]+O(\epsilon^2) = 0, \tag{4.9}$$

$$\bar{u}_t-(\bar{b}\bar{h})_\xi+\frac{\epsilon}{\bar{b}^2}[\kappa\bar{u}\bar{u}_\xi+\bar{b}^2\bar{h}_\xi\tilde{T}(\bar{b}\bar{h}_\xi)]+O(\epsilon^2) = 0. \tag{4.10}$$

It is also possible to derive a single equation for $\bar{h}$ by combining (4.9) and (4.10). To show this, we first multiply (4.9) by $\bar{b}^2$ and then operate with T on the resulting equation. It leads, after iterating with respect to $\bar{u}$, to

$$\begin{aligned}\kappa\bar{u} &= T(\bar{b}^2\bar{h}_t)+\kappa\epsilon[T(\bar{u}\bar{h})_\xi+\bar{h}\tilde{T}\bar{u}_\xi]+O(\epsilon^2)\\ &= T(\bar{b}^2\bar{h}_t)+\epsilon[T\{\bar{h}T(\bar{b}^2\bar{h}_t)\}_\xi+\bar{h}(\bar{b}^2\bar{h}_t)_\xi]+O(\epsilon^2).\end{aligned} \tag{4.11}$$

Operating with $\tilde{T}$ on (4.10) and substituting (4.11), we arrive at the desired equation for $\bar{h}$:

$$\bar{b}^2\bar{h}_{tt}-\kappa\tilde{T}(\bar{b}\bar{h})_\xi+\epsilon\left[\kappa\bar{b}\bar{h}\bar{h}_\xi+\bar{h}_t\,T(\bar{b}^2\bar{h}_t)+\tfrac{1}{2}\tilde{T}(\bar{b}\bar{h}_t)^2+\kappa\tilde{T}\{\bar{h}\tilde{T}(\bar{b}\bar{h}_\xi)\}\right.$$
$$\left.+\tfrac{1}{2}\tilde{T}\left\{\frac{T(\bar{b}^2\bar{h}_t)}{\bar{b}}\right\}^2\right]_\xi+O(\epsilon^2) = 0. \tag{4.12}$$

Here, $\bar{h}_{tt}$ has been replaced in the $O(\epsilon)$ terms by the approximate equation $\bar{h}_{tt} = (\kappa/\bar{b}^2)\,\tilde{T}(\bar{b}\bar{h})_\xi+O(\epsilon)$.

4.2. *Equations with original variables*

In order to transform equations (4.9), (4.10) and (4.12) into those with original physical variables, we first introduce the horizontal component of the surface velocity by

$$u = \phi_x|_{y=\alpha h}. \tag{4.13}$$

Evaluating the relation $\phi_\xi = x_\xi\phi_x+y_\xi\phi_y$ on the free surface, we obtain

$$\bar{u} = x_\xi u+O(\alpha^2), \tag{4.14}$$

where the approximate expression (4.6) has been used together with the definition of $\bar{u}$. We employ (2.8) to calculate x_ξ on $\eta = \alpha\bar{h}$. The result is expressed in the form

$$x_\xi|_{\eta=\alpha\bar{h}} = -\bar{b}+\frac{\pi}{4}\int_{-\infty}^{\infty}\frac{\bar{b}(\xi-\delta\xi')-\bar{b}(\xi)}{\sinh^2(\pi\xi'/2)}\,\mathrm{d}\xi'+O(\alpha^2)$$
$$= -\bar{b}+\frac{\delta^2}{3}\bar{b}_{\xi\xi}+\frac{\delta^4}{45}\bar{b}_{\xi\xi\xi\xi}+\ldots+O(\alpha^2), \tag{4.15}$$

where, in the second line, integrations have been performed after expanding $\bar{b}(\xi-\delta\xi')$ with respect to $\delta\xi'$. Then following the procedure due to Radder (1992), we define the function $\lambda = \lambda(x)$ by

$$x_\xi|_{\eta=\alpha\bar{h}} = -(1+\lambda)\,b. \tag{4.16}$$

Introducing (4.16) into (4.15) yields a nonlinear equation for λ and it can be solved by iteration. The result is

$$\lambda = -\tfrac{1}{3}\delta^2(b_x^2+bb_{xx})+O(\alpha^2)$$
$$= -\tfrac{1}{3}\epsilon^2\{B'^2+(-1+B)B''\}+O(\alpha^2). \tag{4.17}$$

Thus we obtain, within the approximation considered here, the expression

$$x_\xi|_{\eta=\alpha\bar{h}} = -b(x)+O(\alpha^2) = 1-B(\alpha x)+O(\alpha^2), \tag{4.18}$$

and hence

$$\bar{u} = -bu+O(\alpha^2). \tag{4.19}$$

On the other hand, in view of (2.16) the surface elevation $\bar{h}$ can be represented in terms of h as

$$\bar{h} = -h/b+O(\alpha^2). \tag{4.20}$$

It also follows from (2.10), (2.11) and (4.18) that

$$\xi = -\int_0^x \frac{\mathrm{d}x'}{b(x')}+O(\alpha^2) \quad \text{on} \quad \eta = \alpha\bar{h}. \tag{4.21}$$

Finally using (2.8) and (4.18), the ξ-derivative on the surface can be rewritten in the x-derivative as

$$\frac{\partial}{\partial\xi} = (x_\xi+\alpha\bar{h}_\xi x_\eta)|_{\eta=\alpha\bar{h}}\frac{\partial}{\partial x} = -b\frac{\partial}{\partial x}+O(\alpha^2). \tag{4.22}$$

If we substitute (4.19)–(4.22) into (4.9), (4.10) and (4.12), we finally obtain the equations expressed in terms of the original variables:

$$bh_t+\kappa\tilde{T}_b u+\kappa\epsilon[b(uh)_x+\tilde{T}_b\{(h/b^2)\,\tilde{T}_b(bu_x)\}]+O(\epsilon^2) = 0, \tag{4.23}$$

$$u_t+h_x+\epsilon[\kappa uu_x+(1/b)\,h_x\,\tilde{T}_b\,h_x]+O(\epsilon^2) = 0, \tag{4.24}$$

$$bh_{tt}-\kappa\tilde{T}_b\,h_x-\epsilon b[\kappa hh_x+(1/b)\,h_t\,T_b\,h_t+\tfrac{1}{2}\tilde{T}_b(h_t^2/b)+\kappa\tilde{T}_b[(h/b^2)\,\tilde{T}_b\,h_x]$$
$$+\tfrac{1}{2}\tilde{T}_b\{(T_b\,h_t)^2/b^3\}]_x+O(\epsilon^2) = 0. \tag{4.25}$$

Here the integral operators T_b and $\tilde{T}_b$ are defined by

$$T_b\,u(x,t) = -\frac{1}{2\delta}P\int_{-\infty}^{\infty}\coth\left[\frac{\pi}{2\delta}\int_x^{x'}\frac{\mathrm{d}x''}{b(x'')}\right]u(x',t)\,\mathrm{d}x', \tag{4.26a}$$

$$\tilde{T}_b\,u(x,t) = \frac{1}{2\delta}P\int_{-\infty}^{\infty}\frac{u(x',t)}{\sinh\left[\dfrac{\pi}{2\delta}\displaystyle\int_x^{x'}\frac{\mathrm{d}x''}{b(x'')}\right]}\,\mathrm{d}x'. \tag{4.26b}$$

Throughout the paper, the bottom topography has been assumed to be slowly varying with a characteristic lengthscale of $O(\alpha^{-1})$. However, if we further assume that the bump on the bottom is small and has the form

$$B(\alpha x) = \alpha \hat{B}(\alpha x), \tag{4.27}$$

equations (4.23)–(4.25) are considerably simplified. Indeed, carrying out the perturbation analysis, one finds that

$$h_t - \kappa \tilde{T} u + \kappa \epsilon [(uh)_x + \tilde{T}(h\tilde{T}u_x) + (1/\delta)(\tilde{\tilde{T}}_b u)_x] + O(\epsilon^2) = 0, \tag{4.28}$$

$$u_t + h_x + \epsilon(\kappa u u_x - h_x \tilde{T} h_x) + O(\epsilon^2) = 0, \tag{4.29}$$

$$h_{tt} + \kappa \tilde{T} h_x + \epsilon[-\kappa h h_x + 2h_t T h_t + \tilde{T} h_t^2 - \kappa \tilde{T}(h\tilde{T}h_x) - (\kappa/\delta)\tilde{\tilde{T}}_b h_x]_x + O(\epsilon^2) = 0, \tag{4.30}$$

where the operator $\tilde{\tilde{T}}_b$ is defined by

$$\tilde{\tilde{T}}_b u(x,t) = \frac{1}{2\delta} p \int_{-\infty}^{\infty} \mathrm{d}x' \frac{u(x',t)}{\sinh(\pi/2\delta)(x'-x)} \int_{x'}^{x} \hat{B}(\alpha x'')\,\mathrm{d}x'', \tag{4.31}$$

and in deriving (4.30) we have used the formula

$$\tilde{T}(fg) = \tilde{T}[(\tilde{T}f)(\tilde{T}g)] + f\tilde{T}g + g\tilde{T}f. \tag{4.32}$$

When the bottom of the fluid is flat or $b = -1$, T_b and $\tilde{T}_b$ reduce to T and $\tilde{T}$ given respectively by (3.3*b*) and (4.3*b*), and (4.23)–(4.25) and (4.28)–(4.30) reduce to those corresponding to the flat case already derived by Matsuno (1992).

5. Shallow- and deep-water limits

The approximate equations obtained in §4 can be used to describe various wave phenomena over a wide range of fluid depth. However, since the effect of an uneven bottom would be primarily important in shallow water, we first discuss the shallow-water limit of the equations and compare the results with existing theories. After that we briefly comment on the deep-water limit.

5.1. *Shallow-water limit*

In the shallow-water limit $\delta \to 0$, we employ the Boussinesq approximation, namely the parameters α and δ are assumed to be small but finite and they are connected to each other by the relation $\alpha = O(\delta^2)$. To derive approximate equations correct up to $O(\alpha, \delta^2)$, we first expand the operators T_b and $\tilde{T}_b$ defined by (4.26) in powers of δ as

$$T_b f(x) = \frac{1}{2\delta} \int_{-\infty}^{\infty} \operatorname{sgn}(x'-x) f(x')\,\mathrm{d}x' + O(\delta), \tag{5.1a}$$

$$\tilde{T}_b f(x) = -\delta b^2 f_x - \tfrac{1}{3}\delta^3 b^4 f_{xxx} - \delta b b_x f + O(\delta^5), \tag{5.1b}$$

where $\operatorname{sgn}(x-x')$ is the sign function. Note that $b_x = \alpha B'(\alpha x) = O(\alpha) = O(\delta^2)$. In the shallow-water limit, the parameter κ becomes δ^{-1}. Taking this fact and the relation $\epsilon = \alpha\delta$ into account, (4.23)–(4.25) reduce, after substituting (5.1) into them and retaining the terms up to and including $O(\alpha, \delta^2)$, to the equations

$$h_t - (bu)_x - \tfrac{1}{3}\delta^2 b^3 u_{xxx} + \alpha(uh)_x + O(\alpha\delta^2, \delta^4) = 0, \tag{5.2}$$

$$u_t + h_x + \alpha u u_x + O(\alpha\delta^2) = 0, \tag{5.3}$$

$$h_{tt} + (bh_x)_x + \tfrac{1}{3}\delta^2 b^3 h_{xxxx} - \alpha\left[h h_x + \frac{h_t}{b}\int_{-\infty}^{\infty} \operatorname{sgn}(x'-x)\, h_t(x'-x,t)\,\mathrm{d}x'\right]_x + O(\alpha\delta^2, \delta^4) = 0. \tag{5.4}$$

In the case of the flat bottom $b = -1$, these equations coincide with those obtained by Matsuno (1992).

Broer, van Groesen & Timmers (1976) have developed a Hamiltonian method for long water waves over a bottom profile of small slope. They obtained an explicit Hamiltonian under the same assumptions as those employed here (see equation (7.11) in their paper). In terms of our notation, it can be written in the form

$$\mathscr{H} = \int_{-\infty}^{\infty}\left[-\tfrac{1}{2}b\phi_x^2+\tfrac{1}{2}h^2+\frac{\delta^2}{6}b^3\phi_{xx}^2+\frac{\alpha}{2}h\phi_x^2+O(\alpha\delta^2,\delta^4)\right]\mathrm{d}x, \tag{5.5}$$

where the derivatives of the velocity potential are evaluated on the free surface $y = \alpha h$. Now, Hamilton's equations of motion, $h_t = \delta\mathscr{H}/\delta\phi$ and $\phi_t = -\delta\mathscr{H}/\delta h$ yield a system of NEEs for h and ϕ:

$$h_t = (b\phi_x)_x+\tfrac{1}{3}\delta^2(b^3\phi_{xx})_{xx}-\alpha(h\phi_x)_x+O(\alpha\delta^2,\delta^4), \tag{5.6}$$

$$\phi_t = -h-\tfrac{1}{2}\alpha\phi_x^2+O(\alpha\delta^2,\delta^4). \tag{5.7}$$

If we use (2.2) and (4.13) and note $b_x = O(\alpha)$, we find that (5.6) and the x-differentiation of (5.7) coincide with (5.2) and (5.3), respectively within the approximation up to $O(\alpha,\delta^2)$. Since (5.2) and (5.3) have been obtained as special cases of (4.23) and (4.24), respectively, it is natural to suspect whether the latter equations are of Hamiltonian type. This question is, however, not solved as yet.

Equation (5.4) is an analogue of the Boussinesq equation in shallow-water theory and it describes nonlinear waves propagating in both the right and left directions. In order to obtain an equation describing a unidirectional motion to the right for instance, it is appropriate to introduce a new coordinate system according to

$$X = \int_0^x \frac{\mathrm{d}x'}{(-b(x'))^{\frac{1}{2}}}-t, \quad T = \alpha t. \tag{5.8}$$

Then, equation (5.4) reduces to a variable-coefficient Korteweg–de Vries equation (Kakutani 1971)

$$h_T+\frac{b_X}{4\alpha b}h-\frac{\delta^2}{6\alpha}bh_{XXX}-\frac{3}{2b}hh_X+O(\alpha,\delta^2) = 0. \tag{5.9}$$

Note, in this equation, that $b_X = \alpha b_T$ due to $b_t = 0$.

Equation (5.9) is appropriate for solving the initial value problem. Another coordinate transformation is possible which is convenient for treating the boundary value problem. Explicitly it may be written in the form

$$X = \int_0^x \frac{\mathrm{d}x'}{(-b(x'))^{\frac{1}{2}}}-t, \quad Y = \alpha x. \tag{5.10}$$

The equation corresponding to (5.9) now takes the form (Kakutani 1971; Johnson 1973)

$$h_Y+\frac{b_Y}{4b}h+\frac{\delta^2}{6\alpha}(-b)^{\frac{1}{2}}h_{XXX}+\frac{3}{2(-b)^{\frac{3}{2}}}hh_X+O(\alpha,\delta^2) = 0. \tag{5.11}$$

A further reduction can be made if we define the new variables $\hat{h}$, τ and z by (Ono 1972)

$$h = \frac{2\delta^2}{3\alpha}b^2\hat{h}, \quad \tau = \frac{\delta^2}{6\alpha}\int_{Y_0}^{Y}\{-b(Y')\}^{\frac{1}{2}}\mathrm{d}Y', \quad z = X. \tag{5.12}$$

Equation (5.11) then becomes

$$\hat{h}_\tau + 6\hat{h}\hat{h}_z + \hat{h}_{zzz} + \nu\hat{h} + O(\alpha, \delta^2) = 0, \tag{5.13a}$$

where
$$\nu = \nu(\tau) = \tfrac{9}{4}b_\tau/b, \tag{5.13b}$$

represents the effect of an uneven bottom.

In the same way, by using the expansion $\tilde{\tilde{T}}_b f = -\delta\hat{B}f + O(\delta^3)$ together with (5.1) we can show that (4.28)–(4.30) reduce to the following equations:

$$h_t + (1-\alpha\hat{B})\,u_x + \tfrac{1}{3}\delta^2 u_{xxx} + \alpha(uh)_x + O(\alpha\delta^2, \delta^4) = 0, \tag{5.14}$$

$$u_t + h_x + \alpha u u_x + O(\alpha\delta^2) = 0, \tag{5.15}$$

$$h_{tt} - h_{xx} - \tfrac{1}{3}\delta^2 h_{xxxx} + \alpha\left[-hh_x + \hat{B}h_x + h_t\int_{-\infty}^{\infty}\operatorname{sgn}(x'-x)\,h_t(x',t)\,\mathrm{d}x'\right]_x + O(\alpha\delta^2, \delta^4) = 0. \tag{5.16}$$

For the purpose of comparing these equations with existing ones, we introduce the layer-mean horizontal velocity by

$$U = \frac{1}{1+\alpha(h-\hat{B})}\int_{-1+\alpha\hat{B}}^{\alpha h} \phi_x(x,y,t)\,\mathrm{d}y. \tag{5.17}$$

After some calculations, we find that U is related to u by the relation

$$u = U - \tfrac{1}{3}\delta^2 U_{xx} + O(\delta^4). \tag{5.18}$$

Substituting (5.18) into (5.14) and (5.15) and using the approximation $\hat{B}U_x = (\hat{B}U)_x + O(\alpha)$, they are recast in the forms

$$h_t + [(1-\alpha\hat{B}+\alpha h)\,U]_x + O(\alpha\delta^2, \delta^4) = 0, \tag{5.19}$$

$$U_t + h_x + \alpha U U_x - \tfrac{1}{3}\delta^2 U_{xxt} + O(\alpha\delta^2) = 0. \tag{5.20}$$

This system of equations is in agreement with that derived by Lee, Yates & Wu (1989) in the specific case that the surface pressure is constant and the bottom profile is independent of t in their equations.

5.2. *Deep-water limit*

Next we shall investigate NEEs resulting from the deep-water limit $\delta \to \infty$. In this case it is appropriate to rescale the vertical coordinate as $y \to y/\delta$ before taking the limit. If we consider a far-field region of the flow for which $\epsilon x = O(1)$, the bottom profile may be expanded in inverse powers of δ as

$$b(x) = -1 + B(\epsilon x/\delta) = -1 + (\epsilon x/\delta)\,B'(0) + O(\delta^{-2}), \tag{5.21}$$

where we have used $B(0) = 0$. Then (4.23)–(4.25) reduce to

$$h_t + Hu + \epsilon[(uh)_x + H(hHu_x)] + O(\epsilon^2) = 0, \tag{5.22}$$

$$u_t + h_x + \epsilon(uu_x + h_x Hh_x) + O(\epsilon^2) = 0, \tag{5.23}$$

$$h_{tt} - Hh_x - \epsilon[hh_x + H(hHh_x) + H(Hh_t)^2]_x + O(\epsilon^2) = 0. \tag{5.24}$$

Here the operator H is the Hilbert transform defined by

$$Hh(x,t) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{h(x',t)}{x'-x}\,\mathrm{d}x', \tag{5.25}$$

and in deriving (5.24) use has been made of the formula

$$H(fg) = H\{(Hf)(Hg)\} + fHg + gHf, \tag{5.26}$$

which is the deep-water analogue of (4.32). The above equations coincide perfectly with those obtained by Matsuno (1992) in the case of infinite and uniform fluid depth. Thus we find that in the deep-water limit an uneven bottom has no effect on the evolution of surface gravity waves, which is in accordance with physical intuition.

In this paper, we have been concerned with approximate equations correct up to $O(\epsilon)$. However, because of current interest in deep-water phenomena such as higher-order modulation effects (Dysthe 1979; Su 1982; Lo & Mei 1985; Brinch-Nielsen & Jonsson 1986; Akylas 1989, 1991) and instabilities (Yuen & Lake 1982), it is meaningful to take the approximation further. In the case of fluid with infinite depth this can be easily performed. For reference NEEs correct up to $O(\epsilon^2)$ are cited in the Appendix.

6. Conclusions

In this paper we have presented approximate NEEs describing nonlinear surface gravity waves on fluid of varying depth. One advantage of our formalism is that the equations are written in terms of original coordinate variables and therefore they have a direct relevance to physical systems. These equations are uniformly valid from shallow water to deep water and have wide applications in various wave phenomena of current interests such as the deformation of a solitary wave climbing a beach, the reflection of waves by a shelf and the evolution of a solitary wave passing over a submerged obstacle, etc.

Although we have restricted our consideration to waves of small but finite amplitude, the NEEs derived here can easily be extended to include higher-order nonlinear effects. In all cases, higher-order linear dispersion effects can be fully incorporated. The generalized equations would be particularly suitable to describe highly nonlinear phenomena such as the highest solitary wave (Miles 1980) and the breaking wave (Peregrine 1983, 1987). From the mathematical point of view, however, the expansion may be valid only in asymptotic mean and it is probable that solutions are not convergent beyond a certain wave steepness, in analogy with the well-known Stokes' expansion for gravity waves (Schwartz 1974; Cokelet 1977). This is a highly delicate mathematical problem and it should be pursued further in detail. In the future work, we shall investigate these important problems on the basis of the model NEEs obtained in this paper.

The author wishes to thank Professor M. Nishioka for continual encouragement.

Appendix. Higher-order NEEs for deep-water waves

In this Appendix, we describe higher-order NEEs for deep-water waves correct up to $O(\epsilon^2)$. The details of the derivation are omitted and only the final results are quoted. Equations corresponding to (5.22)–(5.24) are now written as follows:

$$h_t + Hu + \epsilon[(uh)_x + H(hHu_x)] + \epsilon^2[H\{hH(hHu_x)_x\} + \tfrac{1}{2}H(h^2u_{xx}) + \tfrac{1}{2}(h^2Hu_x)_x] + O(\epsilon^3) = 0, \tag{A 1}$$

$$u_t + h_x + \epsilon(uu_x + h_xHh_x) + \epsilon^2[2h_xH(uu_x) - 2h_xuHu_x + h_xH(hHh_x)_x + hh_xh_{xx}] + O(\epsilon^3) = 0, \tag{A 2}$$

$$h_{tt} = Hh_x + \epsilon F_x + \epsilon^2 G_x + O(\epsilon^3), \tag{A 3}$$

where
$$F = hh_x + H(hHh_x) + H(Hh_t)^2, \tag{A 4}$$
$$G = \tfrac{1}{2}(h^2Hh_x)_x + H\{hH(hHh_x)_x\} + \tfrac{1}{2}H(h^2h_{xx}) - h_x(Hh_t)^2 + 2H\{hH(h_t h_{xt})\} - 2h_t H(hHh_t)_x + 2H(h_x h_t Hh_t). \tag{A 5}$$

As an application of (A 3), we can show with the aid of the formulae
$$H\,\mathrm{e}^{\mathrm{i}kx} = \mathrm{i}\,\mathrm{sgn}\,k\,\mathrm{e}^{\mathrm{i}kx}, \tag{A 6}$$
$$H(xf) = xHf + \int_{-\infty}^{\infty} f\,\mathrm{d}x, \tag{A 7}$$
that it exhibits a steady periodic wave train of the form
$$h = \mathrm{const} + a\cos\xi + \tfrac{1}{2}\epsilon k a^2 \cos 2\xi + \tfrac{3}{8}\epsilon^2 k^2 a^3 \cos 3\xi + O(\epsilon^3) \quad (\xi = kx - \omega t), \tag{A 8}$$
with
$$\omega = k^{\frac{1}{2}}\{1 + \tfrac{1}{2}\epsilon^2 k^2 a^2 + O(\epsilon^3)\} \quad (k > 0). \tag{A 9}$$
The expression (A 8) is a correct form of the first three terms of the Stokes expansion for deep-water gravity waves. This shows the validity of (A 3).

REFERENCES

AKYLAS, T. R. 1989 Highest-order modulation effects on solitary wave envelopes in deep water. *J. Fluid Mech.* **198**, 387–397.

AKYLAS, T. R. 1991 Higher-order modulation effects on solitary wave envelopes in deep water. Part 2. Multi-soliton envelopes. *J. Fluid Mech.* **224**, 417–428.

BRINCH-NIELSEN, U. & JONSSON, I. G. 1986 Fourth order evolution equations and stability analysis for Stokes waves on arbitrary water depth. *Wave Motion* **8**, 455–472.

BROER, L. J. F. 1974 On the Hamiltonian theory of surface waves. *Appl. Sci. Res.* **29**, 430–466.

BROER, L. J. F. 1975 Approximate equations for long water waves. *Appl. Sci. Res.* **31**, 377–395.

BROER, L. J. F., GROESEN, E. W. C. VAN & TIMMERS, J. M. W. 1976 Stable model equations for long water waves. *Appl. Sci. Res.* **32**, 619–636.

BYATT-SMITH, J. G. B. 1971 An integral equation for unsteady surface waves and a comment on the Boussinesq equation. *J. Fluid Mech.* **49**, 625–633.

COKELET, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. R. Soc. Lond.* A **286**, 183–230.

DYSTHE, K. B. 1979 Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. R. Soc. Lond.* A **369**, 105–114.

GRIMSHAW, R. 1970 The solitary wave in water of variable depth. *J. Fluid Mech.* **42**, 639–656.

JOHNSON, R. S. 1973 On the development of a solitary wave moving over an uneven bottom. *Proc. Camb. Phil. Soc.* **73**, 183–203.

KAKUTANI, T. 1971 Effect of an uneven bottom on gravity waves. *J. Phys. Soc. Japan* **30**, 272–276.

LEE, S. J., YATES, G. T. & WU, T. Y. 1989 Experiments and analysis of upstream-advancing solitary waves generated by moving disturbances. *J. Fluid Mech.* **199**, 569–593.

LO, E. & MEI, C. C. 1985 A numerical study of water-wave modulation based on a higher-order nonlinear Schrödinger equation. *J. Fluid Mech.* **150**, 385–416.

MADSEN, O. S. & MEI, C. C. 1969 The transformation of a solitary wave over an uneven bottom. *J. Fluid Mech.* **39**, 781–791.

MATSUNO, Y. 1992 Nonlinear evolutions of surface gravity waves on fluid of finite depth. *Phys. Rev. Lett.* **69**, 609–611.

MEI, C. C. 1989 *The Applied Dynamics of Ocean Surface Waves* (second printing with corrections). World Scientific.

MEI, C. C. & MÉHAUTÉ, B. L. 1966 Note on the equations of long waves over an uneven bottom. *J. Geophys. Res.* **71**, 393–400.

MILES, J. W. 1980 Solitary waves. *Ann. Rev. Fluid Mech.* **12**, 11–43.

ONO, H. 1972 Wave propagation in an inhomogeneous anharmonic lattice. *J. Phys. Soc. Japan* **32**, 332–336.

PEREGRINE, D. H. 1967 Long waves on a beach. *J. Fluid Mech.* **27**, 815–827.

PEREGRINE, D. H. 1983 Breaking waves on beaches. *Ann. Rev. Fluid Mech.* **15**, 149–178.

PEREGRINE, D. H. 1987 Recent developments in the modelling of unsteady and breaking water waves. In *Nonlinear Water Waves (IUTAM Symp. Tokyo/Japan August 25–28, 1987)* (ed. K. Horikawa & H. Maruo), pp. 17–27. Springer.

RADDER, A. C. 1992 An explicit Hamiltonian formulation of surface waves in water of finite depth. *J. Fluid Mech.* **237**, 435–455.

SCHWARTZ, L. W. 1974 Computer extension and analytic continuation of Stokes' expansion for gravity waves. *J. Fluid Mech.* **62**, 553–578.

STOKES, G. G. 1849 On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.* **8**, 441–455.

SU, M. Y. 1982 Evolution of groups of gravity waves with moderate to high steepness. *Phys. Fluids* **25**, 2167–2174.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves.* Wiley.

WOODS, L. C. 1961 *The Theory of Subsonic Plane Flow.* Cambridge University Press.

WU, T. Y. 1981 Long waves in ocean and coastal waters. *J. Engng Mech. Div. ASCE* **107**, 501–522.

YUEN, H. C. & LAKE, M. M. 1982 Nonlinear dynamics of deep-water gravity waves. *Adv. Appl. Mech.* **22**, 67–229.